

## Midterm

The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. Total point 220 pt.

### Problem 1 : (20 pt)

Let  $G$  be a group and assume that for all  $g \in G$  we have  $g^{-1} = g$ . Prove that  $G$  is abelian.

### Problem 2 : (40 pt)

In  $S_4$ , consider the subset

$$H = \{I_d, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\}$$

1. Compute the inverses of the elements  $H$ .
2. Is  $H$  a subgroup of  $G$ ? Justify your answer.

### Problem 3 : (20 pt)

Let  $H = \{\sigma \in S_n : \sigma(1) = 1\}$ . Prove that  $H$  is not a normal subgroup of  $S_n$  for  $n \geq 3$ .

### Problem 4 : (60 pt)

Let  $G$  be the group of all polynomials of degree  $n$  with real coefficients, that is

$$G = \{P(x) : P(x) = a_0 + a_1x + \cdots + a_nx^n, a_i \in \mathbb{R}\}$$

with addition operation defined by : for  $P(x) = a_0 + a_1x + \cdots + a_nx^n$  and  $Q(x) = b_0 + b_1x + \cdots + b_nx^n \in G$

$$P(x) + Q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

Let  $\phi : (G, +) \rightarrow (\mathbb{R}, +)$  be the evaluation homomorphism given by

$$\phi(P(x)) = P(0)$$

1. (30 pt) Prove that  $\phi$  is homomorphism and that it is surjective but not injective.
2. (10 pt) Find the kernel of  $\phi$ .
3. (20 pt) Prove that the induced map

$$\begin{aligned} \bar{\phi} : (G/\text{Ker}(\phi), +) &\rightarrow (\mathbb{R}, +) \\ [P(x)] &\mapsto P(0) \end{aligned}$$

is a well defined isomorphism.

**Problem 5 : (80 pt)**

Let  $G = \mathbb{Z}/n\mathbb{Z}$  ( $n$  is a fixed integer; if  $x$  is any integer,  $[x]$  is the class of  $x$  modulo  $n$ ).

1. (30 pt) If  $a \in \mathbb{Z}$ , prove that  $\phi_a : G \rightarrow G$  defined by

$$\phi_a([x]) = [ax]$$

is a well defined, group homomorphism and depends only on  $[a]$ .

2. (20 pt) Prove that all homomorphisms  $G \rightarrow G$  are of the type of the previous question. (Hint : Prove that it is enough to fix the image of a cyclic generator of  $G$  to determine fully such a homomorphism.)
3. (20 pt) Prove that  $\phi_a$  is an automorphism if and only if there exists  $b$  such that  $ab \equiv 1 \pmod{n}$ .
4. (10 pt) Prove that if  $n$  is prime, then  $|\text{Aut}(\mathbb{Z}/n\mathbb{Z})| = n - 1$ .